THE SCALING LIMIT OF POLYMER DYNAMICS IN THE PINNED PHASE

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Abstract. We consider the stochastic evolution of a $(1+1)$ dimensional interface (or polymer) in presence of an attractive substrate. We start from a configuration far from equilibrium: an interface with a non-trivial macroscopic profile, and look at the evolution of the space rescaled interface. In the case of infinite pinning force we are able to prove that on the diffusive scale (space rescaled by L in both dimensions and time rescaled by L^2 where L denotes the length of the interface), the scaling limit of the evolution is given by a free-boundary problem with contracting boundaries which belongs to the family of Stefan problems. This is in contrast with what happens for interface dynamics with no constraint or with repelling substrate. We complement our result by giving a conjecture for the whole pinned phase.
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1. Introduction of the model

Random polymer models have been used for a long time used by physicists to model a large variety of physical phenomena. Among the variety of models that have been introduced by theoretical physicists and rigorously studied by mathematicians (see e.g. [9] for a survey of the most studied polymer models), the polymer pinning model, that involves a simple random walk interacting with a defect line, has focused a lot of interest both on the mathematical and on the physical side. The phase transition phenomenon between a pinned phase and a depinned one is now well understood, even in presence of disorder (see [7, 8] and references therein).

On the other hand, dynamical pinning (which has some importance in biophysical application see [1, 2] and references therein) has attracted attention of mathematicians only more recently and a lot of questions concerning relaxation to equilibrium and its connection properties are still unsolved.

Whereas most of the studies up to now (see [3, 4]) has been focused on studying mixing time and relaxation time for the dynamics, we chose here to study here a different aspect, that dynamical scaling limit of the interface. Our aim is to show that the nature of the limit of the rescaled process depends only on whether one lies in the pinned or depinned phase.

In this paper we particularly focus on the model with infinite pinning force and prove that the scaling limit in that case is given by the solution of a free-boundary problem. We complement this result with a conjecture concerning the whole pinned phase. We also give the scaling limit in some region of the unpinned phase: in that case, the interaction with the wall disappears in the limit.

1.1. **A simple model for interface motion with no constraint.** Let us first introduce the simplest version of the model where no substrate is present: the so-called corner-flip dynamics. Let $\Omega = \Omega_L$ denote the set of all lattice paths (polymers) starting at 0 and ending at 0 after $2L$ steps

$$\Omega_L^0 := \{\eta \in \mathbb{Z}^{2L+1} \mid \eta_{-L} = \eta_L = 0\,,\ |\eta_{x+1} - \eta_x| = 1,\ x = -L, \dots, L-1\}\,. \tag{1.1}$$

The stochastic dynamics is defined by the natural spin-flip continuous time Markov chain with state space Ω_L^0. Namely, sites $x = -L, \dots, L$ are equipped with independent Poisson clocks with rate one: when a clock rings at x the path η is replaced by $\eta^{(x)}$, defined by $\eta^{(x)}_y = \eta^{(x)}_x$ for all $y \neq x$ and

$$\eta^{(x)}_x := \begin{cases} \eta_x + 2 \text{ if } \eta_{x\pm1} = \eta_x + 1, \\ \eta_x - 2 \text{ if } \eta_{x\pm1} = \eta_x - 1, \\ \eta_x \text{ if } |\eta_{x+1} - \eta_{x-1}| = 2. \end{cases} \tag{1.2}$$

One denotes by $\widetilde{\eta}(\cdot, t)$, $t \geqslant 0$, the trajectory of the Markov chain, and one considers $\widetilde{\eta}(\cdot, t)$ as a function of the real variable $x \in [-L, L]$ by doing linear interpolation between the integer values of x.

The unique invariant measure for this dynamics is the uniform measure on Ω_L^0, and thus, from standard properties of the random walk, when the system is at equilibrium, the rescaled interface $(\eta(Lx)/L)_{x\in[-1,1]}$ is macroscopically flat. For this model, relaxation to equilibrium is well understood both in terms of mixing-time (see Wilson [15]) or scaling limits (see [11, Theorem 3.2], weaker versions of these result had been known before, using connection with the one dimensional simple symmetric exclusion process).

We cite in full the result concerning the scaling limit for two reasons: it gives some point of reference to better understand results in presence of interface, and we use it as a fundamental building brick for our proof.

Given f_0 a Lipshitz function in $[-1, 1]$, with $f_0(-1) = f_0(1) = 0$, set $\widetilde{f}$ defined on $[-1, 1] \times [0, \infty)$ to be the (unique) solution of the following Cauchy problem,

$$\begin{cases} \partial_t \widetilde{f} &= \Delta \widetilde{f}, \\ \widetilde{f}(\cdot, 0) &= f_0, \\ \widetilde{f}(-1, t) &= \widetilde{f}(1, t) = 0, \quad \forall t > 0. \end{cases} \tag{1.3}$$

We say that an event A_L (or more properly, a sequence of event) occurs *with high probability* if the probability of A_L tends to one when L tends to infinity.

Theorem 1.1. [11, Theorem 3.2] *Let $\widetilde{\eta}^L$ be the dynamic described above with a sequence of initial condition η_0^L satisfying*

$$\eta_0^L(x) = L(f_0(x/L) + o(1)) \text{ uniformly in } x \text{ when } L \to \infty\ . \tag{1.4}$$

Then the space time rescaled version of $\widetilde{\eta}^L$ converges to $\widetilde{f}$ in law for the uniform topology in the sense that for any $T > 0$ with high probability

$$\lim_{L\to\infty} \sup_{x\in[-1,1], t\in[0,T]} \left| \frac{1}{L} \widetilde{\eta}^L(Lx, L^2 t) - \widetilde{f}(x, t) \right| = 0, \tag{1.5}$$

in probability.

The notation in equation (1.4) means that there exists a function ε_L tending to zero such that for all x and L

$$\left|\frac{\eta_0^L(x) - L(f_0(x/L))}{L}\right| \leqslant \varepsilon_L. \tag{1.6}$$

We will keep using this kind of notation in the rest of the paper.

1.2. **Interface interacting with a substrate.** Our main interest in this paper is to understand how this pattern of relaxation to equilibrium is modified (or not modified) when the dynamics has additional constraints. We focus on the case of an interface interacting with a solid substrate and modify our dynamic as follows:

- We consider that a solid wall fills the entire bottom half-plane so that our trajectories have to stay in the positive half-plane ($\eta_x \geqslant 0$, $\forall x \in \{-L, \dots, L\}$).
- The wall interacts with the interface η so that the rates of the transitions that modifies the number of contacts with the wall are changed.

More precisely one starts from a trajectory that lies entirely above the wall, i.e. in the following subset of Ω_L^0:

$$\Omega_L := \{\eta \in \mathbb{Z}^{2L+1} \mid \eta_{-L} = \eta_L = 0\,;\ \forall x \in \{-L, \dots, L-1\},\ |\eta_{x+1} - \eta_x| = 1, \eta_x \geqslant 0\}\,. \tag{1.7}$$

and the rates of the transitions from η to $\eta^{(x)}$ are not uniformly equal to 1 as before but given by

$$c(\eta, \eta^{(x)}) \begin{cases} 0 \text{ if } \eta_x^{(x)} = -1 \quad \text{(interdiction to go through the wall)}, \\ \frac{2\lambda}{1+\lambda} \text{ if } \eta_x = 2 \text{ and } \eta_x^{(x)} = 0, \\ \frac{2}{1+\lambda} \text{ if } \eta_x = 0 \text{ and } \eta_x^{(x)} = 2, \\ 1 \text{ in every other cases }. \end{cases} \tag{1.8}$$

The value of the parameter $\lambda \in [0, \infty]$ determines the nature of the interaction with the wall. If $\lambda > 1$, the transitions adding a contact are favored (which means the wall is attractive), whereas if $\lambda < 1$ the wall is repulsive.

The process defined above is the *heat-bath* dynamics for the *polymer pinning model*, with equilibrium measure $\pi = \pi_{2L}^\lambda$ on Ω defined by

$$\pi_L^\lambda(\eta) = \frac{\lambda^{N(\eta)}}{Z_{2L}^\lambda}\,, \tag{1.9}$$

where $N(\eta) = \#\{x \in [-L+1, L-1] \ :\ \eta_x = 0\}$ denotes the number of zeros in the path $\eta \in \Omega$ and

$$Z_{2L}^\lambda := \sum_{\eta' \in \Omega} \lambda^{N(\eta')} \tag{1.10}$$

is the partition function, namely, the renormalization factor that makes π a probability.

For every $\lambda > 0$, $L \in \mathbb{N}$, $\pi = \pi_L^\lambda$ is the unique reversible invariant measure for the Markov chain. For any value of λ the rescaled version of η at equilibrium (under measure π_L^λ) is flat, but microscopic properties under π_L^λ (for large L) vary with the value of λ:

(i) When $\lambda \in [0, 2)$, the interface is repelled by the wall (i.e. when $\lambda \in (0, 1)$, entropic repulsion takes over energetic attraction of the wall) and typical paths have only a number of contacts with the wall that stays bounded when N tends to infinity,
(ii) When $\lambda \in (2, \infty]$, the interface is pinned to the wall, and typical paths have a number of contacts with the wall which is of order L,

(iii) When $\lambda = 2$, η has a lot of contact with the wall (order $\sqrt{L}$) but the longest excursion away from the wall has length of order L.

For more precise statements and proofs, we refer to Chapter 2 in [7]. These three cases are respectively referred to as the depinned phase, the pinned phase, and the critical point (or phase transition point).

Remark 1.2. The case $\lambda = \infty$, that is our main focus of this paper is a bit particular. Indeed, as seen in (1.8), when $\eta_x = 0$ (when η touches the wall at x), it sticks there forever, so that $\eta(\cdot, t)$ stops moving once one has reached the minimal configuration $\eta^{\min}$ defined by

$$\eta^{\min}_x := \begin{cases} 0 \text{ if } x + L \text{ is even }, \\ 1 \text{ if } x + L \text{ is odd }, \end{cases} \tag{1.11}$$

which is the configuration with the maximal number of contact point with the wall. In that case, the unique invariance probability measure is the Dirac mass on $\eta^{\min}_x$. A question of interest is then to compute the time at which $\eta(\cdot, t)$ stops to move, i.e. the hitting time of $\eta^{\min}$.

Our aim is to get a result similar to Theorem 1.1, describing how, starting from a non-flat profile, the system relaxes to equilibrium. We are able to deduce results in two cases:

- when the wall is strictly repulsive, i.e. when $\lambda \in [0, 1]$: in that case the scaling limit is the same one as for the model without wall (and the result can be obtained by relatively simple comparison).
- when the wall is sticky, i.e. when $\lambda = \infty$: in that case, the attraction of the wall can be seen on the macroscopic level, and the scaling limit is given as the result of a partial differential equation with moving boundary: this is the main result of this paper.

We are also able to present conjectures for all the other cases: we think that the first result extends to the whole depinned phase and at the critical point ($\lambda \in [0, 2]$), whereas in the pinned phase $\lambda > 2$ the scaling limit is roughly the same that in the $\lambda = \infty$ case (see conjecture in Section 1.6).

1.3. **Scaling limit in the repulsive case.** Our first result is an analog of Theorem 1.1 for the dynamic with the wall with repulsion: i.e. in the case $\lambda < 1$.

Theorem 1.3. *Let $\eta = \eta^{L,\lambda}$ be the dynamic on Ω_L with generator $\mathcal{L}$ described above, with the parameter $\lambda \in [0, 1]$ and starting from a sequence of initial condition η^L_0 satisfying*

$$\eta^L_0(x) = L f_0(x/L)(1 + o(1)) \text{ uniformly in } x \text{ when } L \to \infty , \tag{1.12}$$

where f_0 is a 1-Lipshitz non-negative function.

Then $\eta^{L,\lambda}$ converges to $\widetilde{f}$ defined by (1.3) in law for the uniform topology in the sense that for any $T > 0$

$$\lim_{L\to\infty} \sup_{x\in[-1,1], t\in[0,T]} \left| \frac{1}{L}\eta^L(Lx, L^2 t) - \widetilde{f}(x, t) \right| = 0, \tag{1.13}$$

in probability.

This result is not much of a surprise. For this range of λ, the wall is pushing the trajectory η away, so that $\eta(t)$ mostly lies in the wall-free zone: thus the effect of the wall does not appear in the scaling limit. We believe that this also holds in the cases $\lambda \in (1,2)$ and $\lambda = 2$.

1.4. Scaling limit in the attractive case for $\lambda = \infty$, a contracting Stefan problem.

We move now to the case $\lambda = \infty$ (where trajectories cannot move away from the substrate once they have touched it). In that case, it is known that with large probability after a time L^2 the dynamics ends up with a path completely stuck to the substrate (with probability tending to 1), (see [3][Proposition 5.6] and Lemma 2.3 below). This implies in particular that the scaling limit in this case cannot be given by (1.3).

We try now to give some heuristic justification for the scaling limit f. We suppose that the polymer path consists of a pinned region where it sticks to the wall where $f \equiv 0$ and an unpinned region which corresponds to an interval $[Ll(t), Lr(t)]$ (i.e. $(l(t), r(t))$ for the rescaled dynamics) so that $f(t, l(t)) = f(t, r(t)) = 0$. In the unpinned region the wall has no influence and thus Theorem 1.1 indicates that one should have $\partial_t f = \Delta f$, what is left to be determined is the speed at which the boundary of the pinned region moves (value of of the time derivative $l'(t), r'(t)$) and/or the boundary condition that f has to satisfy at the boundary of $(l(t), r(t))$.

Remark 1.4. With the boundary condition that one considers, f is not space derivable at the extremities of $[l(t), r(t)]$. In what follows, when one talk about space derivative of f at point $l(t)$, resp. $r(t)$, we refer to right resp. left derivative.

What we want to justify here is that the slope of the scaling limit at the left resp. right boundary of the pinned region, given by $\partial_x f(l(t), t)$ resp. $\partial_x f(r(t), t)$ has to be equal to $+1$ (resp. -1). The reason is that due to diffusive scaling, for the non-rescaled dynamics, the mean-speed e.g. of the left-boundary of the pinned zone has to be of order $1/L$. This can be achieved only if the density of down-steps near the left boundary is vanishing, and hence if $\partial_x f(l(t), t) = 1$.

Combining this boundary condition with $\partial_t f = \Delta f$, it seems to imply that one must have $l'(t) = -\Delta f(t, l(t))$ and $r'(t) = \Delta f(t, l(t))$.

The main trouble that one gets with this heuristic is that the PDE problem we get out of it, namely

$$\begin{cases} \partial_t f - \Delta f = 0 \in [l(t), r(t)], \\ f(l(t), t) = f(r(t), t) = 0, \quad \partial_x f(l(t), t) = -\partial_x f(r(t), t) = 1, \\ l'(t) = -\Delta f(l(t), t), \quad r'(t) = \Delta f(r(t), t), \quad f(\cdot, 0) = f_0, \end{cases} \tag{1.14}$$

seems ill-posed: there seems to be too much boundary condition etc... However, there exists a solution to the above problem that we can derive from the work of Chayes and Kim in [5]. Unfortunately, this forces us to restrict to the case where the initial condition f_0 is symetric and unimodal (i.e. has a unique maximum).

Given g_0: $[l(0), r(0)] \subset [-1, 1] \to [-1, 1]$ we look at the problem of finding a triplet (g, l, r), such that $g(\cdot, t)$ is defined on $[l(t), r(t)]$ and that

$$\begin{cases} \partial_t g = \Delta g \text{ in } \quad (l(t); r(t)) \,, \\ g(l(t),t) = -g(r(t),t) = 1, \\ l'(t) = -\partial_x g(l(t),t), \quad r'(t) = \partial_x g(r(t),t), \\ g(\cdot,0) = g_0. \end{cases} \tag{1.15}$$

As $l(t)$ and $r(t)$ are moving towards one another, a solution of this problem is defined at most up to the time T where the two boundaries meet. We get the existence of the solution from [5, Theorem 2.2] but it forces us to some restriction on the initial condition.

Proposition 1.5. *Consider g_0 that*

- *$g_0(x) \in [-1,1]$ for all x, $g_0(l(0)) = 1$, $g_0(r(0)) = -1$, and g_0 has smooth derivatives at all order.*
- *g_0 is antisymmetric ($l(0) = -r(0)$ and $g_0(-x) = -g_0(x)$,*
- *g_0 is non-negative on $[l(0), 0]$ and non-positive on $[0, r(0)]$,*

the problem (1.15) admits a classical up to a time $T(g_0)$, for which $l(t)$ and $r(t)$ are C^∞ all the derivative (time and space) are uniformly continuous away from T.

The time T were the solution stops to exist satisfies $l(T) = r(T) = 0$ and it is equal to

$$T(g_0) := \int_{l(0)}^{0} \int_{l(0)}^{t} g_0(t) \, \mathrm{d}t. \tag{1.16}$$

Proof. Consider (ρ, l) the solution of the problem

$$\begin{cases} \partial_t \rho - \Delta \rho = 0 \quad \text{ in } \{(x,t) \mid x \in [l(t), 0]\}, \\ \rho(l(t),t) = 0, l'(t) = \partial_x \rho(l(t),t), \\ \rho(0,t) = 1, \forall t, \quad \rho(x,0) = 1 - g_0(x), \ \forall x \in [l(0), 0]. \end{cases} \tag{1.17}$$

considered in [5] (see equation (1.4) and discussion in Section 2 therein). According to [5, Theorem 2.2], this problem has a unique classical solution up to time $T(g_0)$ (the reader can check that assumptions H1-H2 of the theorem are satisfied in our case).

We set $r(t) = -l(t)$ and define g on $\{(x,t) \mid x \in [l(t), r(t)]\}$ by

$$g(x,t) = \begin{cases} 1 - \rho(x,t), \forall x \in [l(t), 0], \\ \rho(-x,t) - 1, \forall x \in [0, r(t)]. \end{cases} \tag{1.18}$$

Then one can check that by symmetry $g(x,t)$ is infinitely derivable at 0 and is a solution of (1.15).

□

From this we are now ready to describe the solution of (1.14) that we consider in the rest of the paper. Given f_0 a smooth 1-Lipshitz function defined on $[-1,1]$, positive on $(-l(0), r(0))$ and equal to zero elsewhere, set $g_0 := f_0'$ its derivative. Suppose that g_0 satisfies the assumptions of Theorem 1.5 (which means that in particular f is symmetric and unimodal) and set (g, l, r) to be the solution of the Stefan Problem (1.15) given in the proof of Proposition 1.5.

In that case $T = T(g_0) = \int_0^1 f_0(x) \, \mathrm{d}x$. Set

$$f(x,t) := \begin{cases} \int_{l(t)}^{x} g(y,t) \, \mathrm{d}y, \quad & \text{if } t \leqslant T, x \in [l(t), r(t)], \\ 0, \text{ elsewhere }. \end{cases} \tag{1.19}$$

Note that simply by antisymmetry of g, $f(r(t),t)=0$ for all time t so that f is continuous in space for all time. Moreover f is derivable in time except at the points $(t,l(t))$ and $(t,r(t))$ and

$$\begin{aligned}\partial_t f(x,t) = -g(l(t),t)l'(t) + \int_{l(t),x} \Delta g(x,t)\,\mathrm{d}t \\ = \partial_x g(l(t),t) + (\partial_x g(x,t) - \partial_x g(l(t),t)) = \Delta f(x,t). \quad (1.20)\end{aligned}$$

So that it can be said that (f,l,r) is a solution of free-boundary problem (1.14) with initial condition f_0.

Moreover from [5] it can be deduced that this solution of (1.14) is order preserving, in the sense that if one starts with an initial conditions $f_1 \geqslant f_0$ then the solution remains above f for all times.

Theorem 1.6. *Suppose $r(0) = -l(0) = 1$ and let $\eta^{L,\infty}$ be the dynamic with wall and $\lambda = \infty$, and starting from a sequence of initial condition η_0^L satisfying*

$$\eta_0^L(x) = Lf_0(x/L)(1+o(1)) \text{ uniformly in } x \text{ when } L\to\infty\ , \quad (1.21)$$

where f_0 satisfies the above mentioned condition. Set f to be the solution of (1.14) *defined by* (1.19)*.*

Then $\eta^{L,\infty}$ converges to f in law for the uniform topology in the sense that

$$\lim_{L\to\infty} \sup_{x\in[-1,1],t>0} \left| \frac{1}{L}\eta^{L,\infty}(Lx, L^2 t) - f(x,t)\right| = 0, \quad (1.22)$$

in probability. Moreover, one can precisely estimate the time at which the dynamics terminates

$$\mathcal{T} := \inf\{t\geqslant 0 \mid \eta(\cdot,t) = \eta^{\min}\} = L^2 \int_0^1 f_0(x)\,\mathrm{d}x(1+o(1)). \quad (1.23)$$

Remark 1.7. The assumption we have to make concerning f_0 are not crucial for the proof and they only needed to guarantee the existence of a regular solution. We believe that these assumption are purely technical and that classical solution exists with much more generality. The second part of the result (1.23) can be compared to the result obtained in [3], where it was proved that the mixing time for this dynamics was of order L^2 (Theorem 5.5 and Proposition 5.6).

1.5. **Stefan problem and statistical mechanics.** Free boundary problems like (1.15) are called Stefan problems. They appear naturally in thermodynamics to describe the motion of phase boundary in a multiphase medium (e.g. water in solid and liquid state), and for this reason have been the object of extensive studies (see the seminal paper of Stefan [14] and [13] for a survey the subject).

There has been then some efforts in the area of statistical mechanics in order to prove to obtain Stefan problem as a limit microscopic dynamics, among which [6], where Chayes and Swindle have exhibited a particle system whose hydrodynamic limit is given by a Stefan problem, and [12] where Landim and Valle proposed a microscopic modeling of Stefan freezing/melting problem and proved weak convergence of the particle density to the corresponding equation (for a more complete bibliography we refer to the monograph [10]).

The main difference of the problem we study here when compared to the one considered in [6] and [12] is that microscopically the motion of the phase boundary does not depend

only on the state of the system close to the boundary, but on the whole configuration (a contact to the wall can be added anywhere). This makes control of the boundary motion more difficult, and for this reason we did not use an approach based on weak convergence like in most of the literature, but something more based on a more classical interpretation of partial differential equations. For this reason our proof relies on the existence and regularity of a classical solution to the equation that we can obtain for certain class of initial condition thanks to the result in [5].

1.6. **Discussion on the scaling limit in the attractive case for $\lambda \in (2, \infty)$.** Although we are quite far from being able to prove it, we do believe that Theorem 1.6 extends in some way to the localized phase.

We make for simplicity the assumption as before that the polymer is unpinned in a single interval $(l(t), r(t))$, and try to find heuristically the natural boundary condition for the slope of the scaling limit at the extremities of the pinned region $\partial_x f(t, l(t))$, $\partial_x f(t, r(t))$. We obtain it by saying that the system close to the phase separation must be at local-equilibrium.

For precisely we define the free-energy for our pinning model to be

$$\textsc{f}(\lambda) := \lim_{L\to\infty} \frac{1}{2L} \log\left(Z_{2L}^{\lambda}\right). \tag{1.24}$$

Note that this quantity is positive when $\lambda > 2$. And set $d(\lambda) \in (0, 1)$ to be the unique positive solution of

$$\textsc{f}(\lambda) + \frac{1}{2}\log\left(\frac{1-d^2}{4}\right) = 0. \tag{1.25}$$

The value $d(\lambda)$ is the macroscopic slope in the depinned region for a polymer with mixed boundary condition.

Then one must have $\partial_x f(t, l(t)) = d(\lambda) = -\partial_x f(t, r(t))$. Thus altogether the scaling limit f of $\eta^{L,\lambda}(x, t)$ with $\lambda \in (0, 2)$ must satisfy the following free boundary problem

$$\begin{cases} \partial_t f - \Delta f = 0 \in [l(t), r(t)], \\ f(l(t), t) = f(r(t), t) = 0, \quad \partial_x f(l(t), t) = -\partial_x f(r(t), t) = d(\lambda), \\ l'(t) = -\frac{1}{d(\lambda)}\Delta f(l(t), t), \quad r'(t) = \frac{1}{d(\lambda)}\Delta f(r(t), t), \quad f(\cdot, 0) = f_0. \end{cases} \tag{1.26}$$

Note that as $d(\lambda) < 1$ it is possible to have $\Delta f(l(t), t) > 0$ so that contrary to the $\lambda = \infty$ case, the boundaries are not necessarily contracting (and to be more precise, one should precise the rules that are adopted when one reaches the extremities of the interval).

The existence of solutions for (1.26) is not known in general and there are some serious reasons why it cannot be directly deduced from what is done in [5]. However similarly to Proposition 1.5, one can get a solution to (1.26) for a whole class of initial condition ($d(\lambda)$-Lipshitz, symmetric and unimodal).

What is more delicate is to adapt the probabilistic part of the proof of convergence to the scaling limit: maybe the approach that we adopt below in the case $\lambda = \infty$ case could partially adapt (with some non-trivial work) to the case where $\lambda < \infty$, in particular for the lower-bound part 2.4.However, there is a need of completely new idea to prove the upper-bound part of the result as the strategy based on control of the volume (cf. Lemma 2.3) can clearly not adapt directly.

2. Preliminaries

2.1. **Stochastic domination and monotonicity in λ/boundary condition.** Our dynamics has quite enjoyable monotonicity properties that can be proved by standard coupling argument using the so-called *graphical construction*. First introduce a natural order on Ω_0^L. Given for two elements ξ and ξ' we say that $\xi \geqslant \xi'$ if $\xi_x \geqslant \xi'_x$ for every $x \in [-L, L]$. We say that a dynamic η dominates another one η' if one can couple the two dynamic on the same probability space and have $\eta(\cdot, t) \geqslant \eta'(\cdot, t)$ for all t with probability one.

We give some examples of monotonicity that we may use in what follows:

- The dynamic with a wall and $\lambda = 1$ dominates the one without wall.
- If $\lambda < \lambda'$ the dynamic with parameter λ dominates the one with parameter λ'.

A way to construct the coupling is standard (we refer to Section 2.2.1 in [3] where these things are very well explained).

2.2. **A general upper-bound.** Using monotonicity, we prove here that the solution of (1.3) is a general upper-bound for the scaling limit. This provides half of Theorem 1.3, and will be of use for the proof of Theorem 1.6.

Proposition 2.1. *For all choices of $\lambda \in [0, \infty]$, the dynamic starting with initial condition satisfying*

$$\eta_0^L(x) = Lf_0(x/L)(1 + o(1)) \text{ uniformly in } x \text{ when } L \to \infty \;, \tag{2.1}$$

is such that for any given $\varepsilon > 0$ and $T > 0$, w.h.p

$$\frac{1}{L}\eta^L(Lx, L^2 t) \leqslant \widetilde{f}(x, t) + \varepsilon, \forall x \in [-1, 1], t \in [0, T]. \tag{2.2}$$

Proof. We construct an alternative dynamics $\widehat{\eta}$ that constitutes an upper bound for η. In fact $\widehat{\eta}$ follows the same dynamics than η excepts that the updates at distance $L^{3/4}$ from the boundary are rejected and the initial condition is slightly modified so that

- $\widehat{\eta}_0^L(x) = x + L$, for $x \in [-L, -L + 2L^{3/4}]$, $\widehat{\eta}_0^L(x) = L - x$, for $x \in [L - 2L^{3/4}, L]$,
- $\widehat{\eta}_0^L \geqslant \eta_0^L$ and $\widehat{\eta}_0^L(x) \geqslant 2L^{3/4}$, $\forall x \in [-L + 2L^{3/4}, L - 2L^{3/4}]$,
- $\widehat{\eta}_0^L$ satisfies (2.1).

From its initial condition and constraint it follows that $\widehat{\eta}$ is an upper bound for η. Moreover, up to the first time of contact with the wall, in $[-L+L^{3/4}, L-L^{3/4}]$, $\widehat{\eta}$ coincides with a corner-flip dynamics, and as seem in Lemma A.1 in the appendix, with large probability $\widehat{\eta}$ does not touch the wall before time $L^2 T$. Thus we can apply Theorem 1.1 to the corner-flip dynamics on the segment $[-L + L^{3/4}, L - L^{3/4}]$ to get the result. □

The above result is enough to prove Theorem 1.3: it gives one of the two necessary inequalities, the second one can be obtained by noticing that when $\lambda \leqslant 1$ the dynamic is dominates the one without wall, and combining that with Theorem 1.1.

2.3. **Linear decay of the area below the graph of f.** A particular property of f defined by (1.14) is that the area below its graph is decreasing linearly. More precisely, for a function $h : [-1, 1] \to \mathbb{R}_+$ define

$$a(h) := \int_{-1}^{1} h(x)\,\mathrm{d}x. \tag{2.3}$$

One has

Lemma 2.2. *For all $t \in [0, a(f_0)/2]$ one has*

$$a(f(\cdot,t)) = a(f_0) - 2t. \tag{2.4}$$

Proof. It is sufficient to compute the derivative

$$\partial_t a(f(\cdot,t)) = \partial_t \left(\int_{l(t)}^{r(t)} f(x,t)\,\mathrm{d}x \right) \\ = \int_{l(t)}^{r(t)} \partial_x^2 f(x,t)\,\mathrm{d}x = \partial_x f(r(t),t) - \partial_x f(l(t),t) = -2 \tag{2.5}$$

□

2.4. **An upper bound for the decay of the area below the graph of $\bar{\eta}$.** From now on, we focus on the case $\lambda = \infty$ only. Consider $\eta = \eta^{L,\infty}$ to be the realization of our Markov chain and $\bar{\eta}$ to be the rescaled version (defined on $[-1,1]$)

$$\bar{\eta}(x,t) = \frac{1}{L}\eta(Lx, L^2 t). \tag{2.6}$$

Consider $a(\bar{\eta}(\cdot,t))$ the area below this rescaled curve (in what follows we use the notation $\eta(t) = \eta(\cdot,t)$ to denote the state of the Markov chain at tine t). We prove that something similar to Lemma 2.2 holds: that the area below $\bar{\eta}$ decreases at least linearly (apart from some random perturbation of lower order).

To state fully our result, it is handier to consider first the area below the non-rescaled curve:

$$A(\eta(t)) := \int_{-L}^{L} \eta(x,t)\,\mathrm{d}x. \tag{2.7}$$

It can be checked by the reader (see also Section 5.4 in [3]) that the expected drift of $A(\eta(t))$ (recall that a corner flip changes the area by ± 2) is equal to $D(\eta(t))$ where

$$D(\eta) := 2|\{x \in \{-L \dots L\} \mid \eta_x = \eta_{x\pm 1} - 1 \text{ and } \eta_x \neq 0\}| \\ - 2|\{x \in \{-L \dots L\} \mid \eta_x = \eta_{x\pm 1} + 1 \text{ and } \eta_x \neq 1\}| \geqslant -2, \tag{2.8}$$

where the last inequality is valid for all $\eta \neq \eta^{\min}$ (for which the drift is equal to 0).

More precisely $D(\eta)$ is equal to minus the number of excursions of length 4 or more away from the wall. Applying martingale techniques allows to have better control over $A(\eta(t))$, by bounding its variance.

Lemma 2.3. *One has w.h.p,*

$$A(\eta(t)) \leqslant A(\eta(0)) + \int_0^t D(\eta(s))\,\mathrm{d}s + L^{7/4}, \quad \forall t \in [0, L^2], \tag{2.9}$$

so that in particular for all $t \in [0, L^2]$

$$A(\eta(t)) \leqslant \max(A(\eta(t)) - 2t + L^{7/4}, L). \tag{2.10}$$

As a consequence, w.h.p.

$$\mathcal{T} \leqslant A(\eta(0))/2 + L^{7/4}. \tag{2.11}$$

and w.h.p uniformly for all time,

$$a(\bar{\eta}(t)) \leqslant (a(f_0) - 2t)_+ + o(1), \tag{2.12}$$

Proof. One can check that $A(\eta(t))+\int_0^t D(\eta(s))\,\mathrm{d}s$ is a martingale (for the natural filtration associated to $\eta(t)$, $t \geqslant 0$). Its quadratic variation is equal to

$$\int_0^t F(\eta(s))\,\mathrm{d}s \tag{2.13}$$

where

$$F(\eta(s)) := 4\big(|\{x \in \{-L \dots L\} \mid \eta_x = \eta_{x\pm1}+1 \text{ and } \eta_x \neq 0\}| \\ + |\{x \in \{-L \dots L\} \mid \eta_x = \eta_{x\pm1}-1\}|\big) \leqslant 4L. \tag{2.14}$$

Hence the first statement is obtained simply by using Doobs maximal inequality, noticing that at time L^2 the variance is bounded by $4L^3$.

For the second statement, either the chain has already terminated and thus $A(t) = L$ or it has not and thus the first statement gives us the answer as $D(s) \leqslant -1$ in $(0,t)$.

To get (2.11) notice that if one supposes that $\mathcal{T} \geqslant A(\eta(0))/2 + L^{7/4}$, then

$$\int_0^{A(\eta(0))+L^{7/4}} D(\eta(s))\,\mathrm{d}s \leqslant -2A(\eta(0)) - 2L^{7/4}, \tag{2.15}$$

and thus if this happens with non-vanishing probability, one gets a contradiction to (2.9). Equation (2.12) is just the renormalized version of (2.10). □

A consequence of equation (2.12) is that in order to prove Theorem 1.6 one only needs to prove a lower-bound result. Then the upper-bound on the area plus the constraint that $\bar\eta(\cdot,t)$ has to be Lipshitz are sufficient to get the opposite bound. More precisely we need to prove

Proposition 2.4. *Suppose $r(0) = -l(0) = 1$ and let $\eta^{L,\infty}$ be the dynamic with wall and $\lambda = \infty$, and starting from a sequence of initial condition η_0^L satisfying*

$$\eta_0^L(x) = Lf_0(x/L)(1+o(1)) \text{ uniformly in } x \text{ when } L \to \infty\ , \tag{2.16}$$

Then for every choice of $\varepsilon > 0$ the rescaled dynamics $\bar\eta$ satisfies w.h.p.

$$\bar\eta^{L,\infty}(x.t) \geqslant f(x,t) - \varepsilon, \forall x \in [-1,1], \forall t > 0 \tag{2.17}$$

Proof of Theorem 1.6 from Proposition 2.4. It is sufficient to prove that for any $\delta > 0$ w.h.p

$$\bar\eta(x.t) \leqslant f(x,t) - \delta, \quad \forall x \in [-1,1], \forall t > 0. \tag{2.18}$$

By equation (2.12) and Lemma 2.2 one knows that w.h.p

$$a(\bar\eta(t)) \leqslant a(f(\cdot,t)) + \delta^2/32. \tag{2.19}$$

Moreover one has from Proposition 2.4 that w.h.p. for all x and t

$$\bar\eta(x,t) \geqslant f(x,t) - \delta^2/32. \tag{2.20}$$

Hence w.h.p. $\bar\eta(x,t) - f(x,t) + \delta^2/32$ is a 2-Lifshitz positive function whose integral is smaller than $\delta^2/16$ which implies

$$\bar\eta(x,t) - f(x,t) - \delta^2/32 \leqslant \delta/2. \tag{2.21}$$

Concerning (1.23), the upper-bound on $\mathcal{T}$ is proved in Lemma 2.3, and the lower-bound is a consequence of (1.22).

□

3. Proof of Proposition 2.4

3.1. **Overall strategy.** The strategy to prove Proposition 2.4, which is quite inspired by the one adopted in [11] for zero temperature stochastic Ising model, is to iterate infinitesimal statements many time.

However, whereas one can control the motion of the interface away from the boundary for infinitesimal time using Theorem 1.1 and a method similar to the one used in [11], controling the motion of the boundary between the pinned phase and the unpinned one is more difficult. For this reason we could not prove Proposition 2.4 directly. What we do first is add a small perturbation of amplitude δ to the function f and to the initial condition η_0^L (see the caption of Figure 1). The idea of adding this perturbation is that by adding flat parts of slope $+1/-1$ on the side of η. one slows down a lot the motion of the phase separation boundary.

Using the results of the previous section concerning areas, we prove that to prove Proposition 2.4, it is sufficient to prove it with f replaced by the modified version f^δ (Proposition 3.1), And then one reduces Proposition 3.1 to controlling the dynamics only up to a short time ε with Lemmata 3.2 and 3.3. The key point in the proof of these Lemmata, is Proposition 4.1, that controls the motion of the boundaries.

3.2. **Modification of the initial function.** Given f_0 with $l(0) > -1$, $r(0) < -1$, set $\bar\delta = a(f_0)\delta$ and $\bar\delta(t) := \delta a(f(t)) = \delta(a(f_0) - 2t)$ (recall Lemma 2.2) and define $f^\delta : [-1,1] \times (0,\infty) \to \mathbb{R}_+$ (for δ small enough) by

$$f^\delta(x,t) := \begin{cases} f(x,t) + \bar\delta(t), \text{ in } (l(t), r(t)),\\ x - (l(t) - \bar\delta(t), \text{ in } (l(t) - \bar\delta(t), l(t)),\\ -\big(x - (r(t) + \bar\delta(t))\big), \text{ in } (r(t), r(t) + \bar\delta(t)),\\ 0 \quad \text{elsewhere.} \end{cases} \tag{3.1}$$

See Figure 1 for a graphical vision of f^δ.

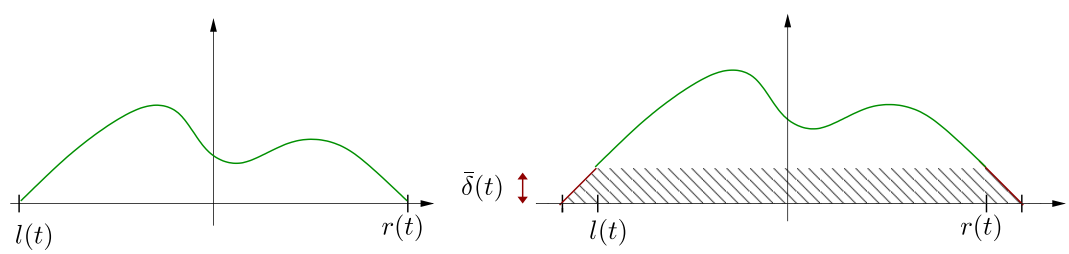

Figure 1. The construction of the graph of $f^\delta(t)$ from the graph of $f(t)$ is done by adding some kind of pedestal of height $\bar\delta(t)$ to support the original graph.

Our aim is to show that if one starts from a configuration that is above f_0^δ then it remains above f^δ for all time.

Proposition 3.1. *Given a f_0 regular enough, starting from a (sequence of) initial condition η_0^L satisfying*

$$\bar\eta_0^L(x) \geqslant f_0^\delta(x)(1 + o(1)) \quad \forall x \in [-1,1]. \tag{3.2}$$

one has for every given $\varepsilon > 0$ w.h.p

$$\bar{\eta}(x,t) \geqslant f^\delta(x,t) - \varepsilon. \tag{3.3}$$

Proposition 3.1 is substantially easier to prove that Proposition 2.4 for reasons that we expose now. Notice that for $x \in [l(t), r(t)]$

$$\partial_t f^\delta(x,t) = \Delta f(x,t) - 2\delta = \Delta f^\delta(x,t) - 2\delta, \tag{3.4}$$

and that out of these interval $\partial_t f^\delta(x,t) = \Delta f(l(t),t) - \delta = \Delta f(r(t),t) - \delta$ (recall that f is symetric). The motion of the boundaries of the support is done at speed $-\Delta f(l(t),t) + \delta$.

However, Theorem 1.1 says that the local push for the scaling limit (away from the wall) should be $\Delta f(x,t)$ in $x \in [l(t), r(t)]$ and 0 on the flat parts on the side, so that we have some extra error-margin to prove the result. Moreover, adding the flat parts on the side helps to control the motion of the boundary.

Proof of Proposition 2.4 from Proposition 3.1. Given f_0, and ε, we consider $\delta > 0$ small enough and define a modification $\widehat{f_0}$ of f_0 that is smooth, 1-Lipshitz and unimodal, and that satisfies

$$\widehat{f_0}^\delta \leqslant f_0, \quad \text{ and } \quad \int_{[-1,1]} (f_0 - \widehat{f_0})(x)\, \mathrm{d}x \leqslant \varepsilon^2/4. \tag{3.5}$$

Now remark that if η_0^L satisfies (2.16) then it has to satisfy (3.2) with f_0^δ replaced by $\widehat{f_0}^\delta$. thus one gets using Proposition 3.1 that for any ε w.h.p. for all $t \geqslant 0$ and $x \in [-1,1]$

$$\bar{\eta}(x,t) \geqslant \widehat{f}^\delta(x,t) - \varepsilon \geqslant \widehat{f}(x,t) - \varepsilon. \tag{3.6}$$

Where $\widehat{f}$ denotes the solution of (1.14) with initial condition $\widehat{f_0}$ and $\widehat{f}^\delta$ is defined as in (3.1), replacing f by $\widehat{f}$.

By Lemma 2.2 and the choice of initial condition, one has for any times.

$$\int_{-1}^{1} (f(x,t) - \widehat{f}(x,t))\, \mathrm{d}x \leqslant \varepsilon^2/4. \tag{3.7}$$

As $(f - \widehat{f})(\cdot, t)$ is $2-$Lipshitz and positive, this implies that for all t and x, $f \leqslant \widehat{f} + \varepsilon$, so that (3.3) implies that

$$\bar{\eta}(x,t) \geqslant f(x,t) - 2\varepsilon. \tag{3.8}$$

□

3.3. **Reduction to a statement for infinitesimal time.** We prove now Proposition (3.1), by showing that this is enough to control the dynamics only for a time ε. The proof of the control of the dynamic for infinitesimal time, formalized below in Lemma 3.2 and 3.3 is postponed to the next Section.

Lemma 3.2. *For any smooth f_0 and given δ, for any initial condition η_0 satisfying* (3.2) *there exists $\varepsilon_0 = \varepsilon_0(\delta, f_0) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, w.h.p*

$$\bar{\eta}(x,\varepsilon) > f^\delta(\varepsilon, x), \forall x \in [-1,1]. \tag{3.9}$$

Moreover, for any given $c > 0$ can be chosen ε_0 so that it works simultaneously for all initial condition of the type $f(\cdot,t)$, for all $t \in [0, a(f_0)/2 - c]$ (on any interval that is separated from the disappearance time).

Lemma 3.3. *For any smooth f and given δ and ε_1, for any initial condition η_0 satisfying* (3.2) *there exists ε_0 (the same as above) such that w.h.p. for all $t \in (0, \varepsilon_0)$,*

$$\bar{\eta}(x,t) > f^{\delta}(x,t) - \varepsilon \tag{3.10}$$

Moreover, for any given $c > 0$ can be chosen ε_0 so that it works simultaneously for all initial condition of the type $f(\cdot,t)$, for all $t \in [0, a(f_0)/2 - c]$ (on any interval that is separated from the disappearance time).

Proof of Proposition 3.1 from Lemmata 3.2 and 3.3. Let us fix c arbitrary small. We consider the dynamics only up to a time c before disappearance. After this time one has

$$a(f(\cdot,t)) \leqslant 2c, \tag{3.11}$$

so that $f(x,t) \leqslant \sqrt{2c}$ uniformly (as this is a 1-Lipshitz function), so that

$$f^{\delta}(x,t) \leqslant \sqrt{2c} + \delta 2c. \tag{3.12}$$

As $\bar{\eta}$ is non-negative, this trivially implies (3.3) if c has been chosen small enough.

Let us fix ε so that Lemma 3.2 is valid starting with initial condition $f(\cdot,t)$ for any $t \in [0, T-c]$. Then by iterating Lemma 3.2 one gets that w.h.p, for all $t = \varepsilon k$, with k such that $\varepsilon k \leqslant T - c$.

$$\bar{\eta}(x,t) \geqslant f^{\delta}(t,x). \tag{3.13}$$

We need now to control the time windows $(\varepsilon k, \varepsilon(k+1))$, for all k such that $\varepsilon k \leqslant T - c$. When $\bar{\eta}(x, \varepsilon k) \geqslant f^{\delta}(t,x)$ (which occurs for every k w.h.p.) one can apply Lemma 3.3 for the chain with initial contion $\bar{\eta}(x, \varepsilon k)$ and get that (3.10) holds w.h.p. for $t \in (\varepsilon k, \varepsilon(k+1))$. On concludes using union bound on k.

□

4. Proof of Lemmata 3.2 and 3.3

The rest of the section is devoted to the proof of Lemma 3.2 and Lemma 3.3. We suppose that f^{δ} has initial support on $[-1,1]$ for notational purpose (if not an armless scaling can bring us back to this case). Moreover, by monotonicity, it is sufficient to prove the result starting from an initial condition satisfying

$$\bar{\eta}_0^L(x) = f_0^{\delta}(x)(1 + o(1)). \tag{4.1}$$

In fact the most difficult thing to prove for that result is that the frontier between the pinned region and the unpinned one does not move to fast. The control of the speed is much facilitated by the addition of the straight-line portion on the sides but remains non-trivial. This problem is treated in the following proposition.

Proposition 4.1. *If ε is chosen small enough (how small depending only on $\bar{\delta} := \delta a(f_0)$), the dynamic started from an initial condition satisfying* (3.2) *satisfies w.h.p.*

$$\bar{\eta}(x,t) > 0, \forall x \in [-1+\varepsilon^2, 1-\varepsilon^2]. \tag{4.2}$$

Proof. The idea is to show that if one touches the wall too soon, we get a contradiction of Lemma 2.3 that concerns the area below the curve $a(\bar{\eta})$. Thus the first step of our reasoning is to show that the area below $\bar{\eta}$ decreases roughly linearly up to the time it touches the wall (and estimate the error term).

Set $d=\varepsilon^2$,$\mathcal{T}_d$ be the time at which η first touches the wall in $[-(1-d)L,(1-d)L]$ and $\tau_d=\mathcal{T}_d/L^2$. Let $A^d(\eta)$ be equal to the area below the curve in the interval $[-(1-d)L,(1-d)L]$ (see Figure 2),

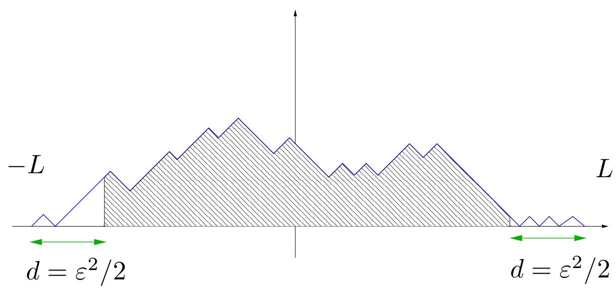

FIGURE 2. A trajectory η, with the volume $A^d(\eta)$ darkened. The drift of the volume when there are no contact with the wall, is equal to the difference of the slope of η at the left and at the right of the interval $[-(1-d)L,(1-d)L]$. It is either equal to 2, 0, or -2.

$$A^d(\eta(t)):=\int_{-L(1-d)}^{L(1-d)}\eta(x,t)\,\mathrm{d}x. \tag{4.3}$$

We use the observation made in that before time $\mathcal{T}_d$, the drift of $A^d(\xi)$ is at least -2 up to time $\mathcal{T}_d$ to get the following result

Lemma 4.2. *One has w.h.p. for all $t\leqslant\mathcal{T}_d$,*

$$A^d(\eta(\cdot,t))\geqslant A^d(\eta(\cdot,0))-2t+O(L^{7/4}), \tag{4.4}$$

and as a consequence, w.h.p.

$$a(\bar\eta(\tau_d))\geqslant a(f_0^\delta)-2\tau_d+d^2+o(1). \tag{4.5}$$

Proof of Lemma 4.2. The general idea is the same that in Lemma 2.3. One start noticing that the average drift of $A^d(\eta(t))$ is at least -2 up to time $\mathcal{T}_d$. Indeed it is equal to $D^d(\eta(t))$ where

$$\begin{aligned}D^d(\eta):=2\big(&|\{x\in[-(1-d)L,(1-d)L]\mid\eta_x>1,\eta_{x\pm1}=\eta_x-1\}|\\&-|\{x\in[-(1-d)L,(1-d)L]\mid\eta_x>0,\eta_{x\pm1}=\eta_x+1\}|\big)\end{aligned} \tag{4.6}$$

which is either equal to 2, 0 or -2.

Then we control the variance of the martingale $A^d(\xi(\cdot,t))-D^d(t)$ (the quadratic version can be shown to be uniformly $O(Lt)$) to get the conclusion using Doobs inequality.

We get the second statement by rescaling and noticing that

$$a(\bar\eta(\tau_d))\leqslant\frac{1}{L^2}A^d(\bar\eta(\mathcal{T}_d))+d^2. \tag{4.7}$$

□

The next step is to find an upper-bound on $a(\bar\eta(\tau_d))$ that holds w.h.p. and contradicts (4.5) if τ_d is too small. For this purposes we combine two statements:

(i) According to Proposition 2.1, one has $\bar\eta\leqslant\widetilde{f}^\delta+o(1)$, where $\widetilde{f}^\delta$ is defined as the solution of (1.3) with initial condition f_0^δ.

(ii) If x_d is the point in $[-(1-d),(1-d)]$ where $\bar\eta$ touches the wall, then as $\bar\eta$ is Lipshitz, $\bar\eta(\tau_d)\leqslant |x-x_d|$.

We need some work to control the area below $g(x):=\min(\widetilde{f}^\delta(x,\tau_d),|x-x_d|)$ (see Figure 3). We are going to prove that for ε small enough (depending on $\bar\delta$) if $\tau_d\leqslant\varepsilon$,

$$a(g)\leqslant a(\widetilde{f}^\delta(\tau_d))-d\bar\delta/8. \tag{4.8}$$

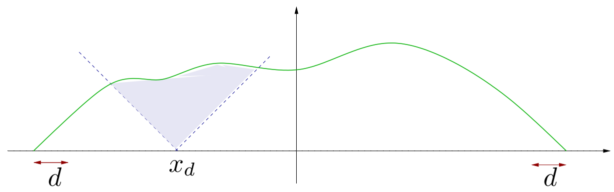

FIGURE 3. Figure representing the two function $\widetilde{f}^\delta(\cdot,\tau_d)$, and $|\cdot-x_d|$. The difference of area between $a(\widetilde{f}^\delta(\tau_d))-a(g)$ is the dark area on the figure. We prove a lower-bound on it using that $\widetilde{f}^\delta$ is not too small if x_d lies in the middle of the interval, or that the slope of f on the boundary is close to one if x_d is closer to the boundary.

By symmetry we only need to consider the case $x_d\leqslant 1/2$. If $x_d\geqslant(-1+\bar\delta/4,1/2)$, one has, thanks to (A.7) Lemma A.2,

$$g(x)=x-x_d\leqslant\widetilde{f}^\delta(\cdot,\tau_d)-\delta/16,\quad\forall x\in(x_d,x_d+\delta/16) \tag{4.9}$$

and hence

$$a(g)\leqslant\ \leqslant a(\widetilde{f}^\delta(\tau_d))-\bar\delta^2/256. \tag{4.10}$$

If $x_d\in[-(1-d),1-\bar\delta/4]$, for all $x\in(x_d,x_d+\delta/4)$, thanks to (A.6) (recall that we suppose $\tau_d\leqslant\varepsilon=d^{1/2}$)

$$g(x)=x-x_d\leqslant\widetilde{f}^\delta(x,\tau_d)-(1+x_d)+\bar\delta\exp(-\bar\delta^2/(16d^{1/2}))\leqslant\widetilde{f}^\delta(x.\tau_d)-d/2. \tag{4.11}$$

So that (4.10) holds.

Thus if $\tau_d\leqslant\varepsilon$ with probability not tending to zero, on has (using (A.5))

$$a(\bar\eta)\leqslant a(\widetilde{f}^\delta)-d\bar\delta/8+o(1)\leqslant a(\widetilde{f}^\delta_0)-2t+2\exp(-\frac{\bar\delta^2}{16t}) \tag{4.12}$$

w.h.p. conditioned on this event. This contradicts Lemma 4.2, and thus one has that with large probability $\tau_d\geqslant\varepsilon$.

□

We just proved that the dynamics does not touch the wall in the interval $[-(1-\varepsilon^2),(1-\varepsilon^2)]$, it allows us to compare it more easily with the dynamic with no-wall for which we have a scaling Theorem. If one runs the dynamic up to a time ε, according to Proposition 4.1 the dynamic $\bar\eta$ coincide w.h.p. with a modified one $\bar\eta^{(\varepsilon)}$ where there is no wall-constraint in the interval $[-(1-\varepsilon^2),(1+\varepsilon^2)]$.

Using monotonicity of the dynamics, this second dynamic can be bounded from below by a dynamic with the domain reduced to $[-(1-\varepsilon^2),(1+\varepsilon^2)]$ and with modified initial condition changed to one that is below the original and that satisfies $\bar\eta^\varepsilon_0=f_0^{(\varepsilon,\delta)}+o(1)$ where $f_0^{(\varepsilon,\delta)}$ is defined on $[-(1-\varepsilon^2),(1-\varepsilon^2)]$ by

$$f_0^{(\varepsilon,\delta)} := f_0^\delta(x) - 2\varepsilon^2 \leqslant \min(f_0^\delta(x), x + (1-\varepsilon^2), (1+\varepsilon^2) - x). \tag{4.13}$$

One calls $\bar\eta^{(\varepsilon)}$ the resulting rescaled dynamics. For this dynamics which is a corner-flip dynamics one can derive the scaling limit from Theorem 1.1 and get that up to time ε uniformly for all x,

$$\bar\eta^{(\varepsilon)}(x,t) = \widetilde{f}^{(\varepsilon,\delta)}(x,t) + o(1), \tag{4.14}$$

where $\widetilde{f}^{(\varepsilon,\delta)}(x,t)$ denotes the solution of equation (1.3) with initial condition $f_0^{(\varepsilon,\delta)}$. Thus it is sufficient to prove Lemma 3.2 and Lemma 3.3 with $\bar\eta$ replaced by $\widetilde{f}^{(\varepsilon,\delta)}$ the solution of (1.3) on the interval $[-(1-\varepsilon^2),(1-\varepsilon^2)]$ starting from initial condition $f_0^{(\varepsilon,\delta)}$. Lemma 3.3 is the easier part and we leave it to the reader and concentrate on Lemma 3.2, and try to prove

$$\widetilde{f}^{(\varepsilon,\delta)}(x,\varepsilon) > f^\delta(x,\varepsilon), \tag{4.15}$$

for all $x \in [l(\varepsilon)+\delta(a(f_0)-2t), r(\varepsilon)-\delta(a(f_0)-2t)]$. Controlling the value of $f^\delta(x,\varepsilon)$ is not too difficult, as one knows from [5, Theorem 2.2], that $\partial_t f$ is regular away from the disappearance time so that looking at the definition of f^δ, we can find a constant C_1 (depending on f_0) such that for all $t \in [0, a(f_0)/2 - c]$,

$$f^\delta(x,t+\varepsilon) \leqslant \begin{cases} f^\delta(x,t) + \varepsilon(\Delta f(x,t) - 2\delta) + C_1\varepsilon^2, \quad \forall x \in (l(t+\varepsilon), r(t+\varepsilon)), \\ f^\delta(x,t) + \varepsilon(\Delta f(l(t),t) - 2\delta) + C_1\varepsilon^2, \\ \quad \forall x \in [l(t+\varepsilon) - \bar{\bar\delta}(t+\varepsilon), l(t+\varepsilon)] \cup [r(t+\varepsilon), r(t+\varepsilon) + \bar{\bar\delta}(t+\varepsilon)]. \end{cases} \tag{4.16}$$

For the second line notice that by symmetry $\Delta f(l(t),t) = \Delta f(r(t),t)$. Moreover

$$l(t\varepsilon) - l(t) = -(r(t+\varepsilon) - r(t)) \geqslant -\varepsilon\Delta f(l(t),t) - C_1\varepsilon^2. \tag{4.17}$$

Controlling $\widetilde{f}^{(\varepsilon,\delta)}(x,\varepsilon)$ is more tedious: as the initial condition is not C^2, there is no continuity of the time derivatives. However one can control things manually (and to justify carefully that the reasoning works for all initial conditions $f^\delta(\cdot,t)$). Note that

$$\widetilde{f}_0^{(\varepsilon,\delta)}(x,\varepsilon) = f^{(\delta,\varepsilon)}(x,0) - \varepsilon^2 + \mathbf{E}_x^*\left[\int_0^\varepsilon \Delta f_0^{(\delta,\varepsilon)}(B_t)\,\mathrm{d}t\right], \tag{4.18}$$

where $\mathbf{E}_x^*$ is the law of a standard Brownian motion starting from x and stopped when it touches the boundaries of the interval $[-(1-2\varepsilon^2),(1-2\varepsilon^2)]$. Then one can check (e.g. by using Taylor Young formula and the fact that Δf_0 is non-positive near $r(t)$ and $l(t)$) that there exists a universal constant C_2 such that,

$$\mathbf{E}_x^*\left[\Delta f_0^\delta(B_t)\right] \geqslant \begin{cases} \Delta f_0^\delta(x) - C_2\|\partial^3_{x^3} f_0\|_\infty\varepsilon \text{ if } x \in (l(0),r(0)), \\ \Delta f_0(l(0)) - C_2\|\partial^3_{x^3} f_0\|_\infty\varepsilon \text{ if } x \in [-1+\varepsilon^2, l(0)] \cup [r(0), 1-\varepsilon^2]. \end{cases} \tag{4.19}$$

Thus one has

$$\widetilde{f}_0^{(\varepsilon,\delta)}(x,\varepsilon) \geqslant \begin{cases} f_0^\delta(x) + \Delta f_0(x)\varepsilon - C_2\|\partial^3_{x^3} f_0\|_\infty\varepsilon^2, \quad \forall x \in (l(0),r(0)), \\ f_0^\delta(x) - \Delta f_0(l(0))\varepsilon - C_2\|\partial^3_{x^3} f_0\|_\infty\varepsilon^2, \forall x \in [-1+\varepsilon^2, l(0)] \cup [r(0), 1-\varepsilon^2]. \end{cases} \tag{4.20}$$

Combining (4.16), (4.17) and (4.20), one gets (4.15) provided that ε is small enough compared to $\bar\delta$. Finally one has to check that this reasoning can be valid simultaneously for all initial condition $f(\cdot,t)$ with $t \in [0,(a(f_0)/2)-c]$: there is no problem for Proposition

4.1 as $\bar\delta \geqslant 2c\delta$ for all choice of t. The constant C_1 in (4.16), (4.17) has been chosen to be valid for all $t \in [0, (a(f_0)/2) - c]$. For (4.20) one notices that the third derivative $\|\partial^3_{x^3} f(\cdot,t)\|_\infty$ is bounded uniformly in $t \in [0, (a(f_0)/2) - c]$.

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Appendix A.

In this Section, we prove several technical results. The first one concerns the time needed for a corner-flip dynamics to reach an atypically low position.

Lemma A.1. *Consider a corner-flip dynamics on Ω_L^0, started from an initial condition that satisfies*

$$\eta_0(x) \geqslant \min(x+L, -x+L, L^{3/4}). \tag{A.1}$$

Then with large probability, for all $t \leqslant \exp(L^{1/4})$

$$\widetilde\eta(x,t) \geqslant -L^{3/4}, \forall x \in [-L, L] \tag{A.2}$$

Proof. One couples $\widetilde\eta$ with a dynamic $\widetilde\eta^2$ starting from the uniform measure on Ω_L^0 (we denote it by π). Because of our choice for the initial condition of $\widetilde\eta$ is above $\widetilde\eta^2$ at time zero with large probability and thus we can couple the two dynamics so that $\widetilde\eta \geqslant \widetilde\eta^2$ for all time with large probability.

Hence it is sufficient to prove the result for $\widetilde\eta^2$. Consider the discrete-time dynamics $\widehat\eta^2(n)$ starting from the uniform measure and that at each step choses x at random and flip η to $\eta^{(x)}$. As π is left invariant by this dynamics, the probability that $\widehat\eta^2(x,n) < -L^{3/4}$ for some x after $\exp(2L^{1/4})$ step is at most (by union bound)

$$\exp(2L^{1/4})\pi(\exists x \in [-L, L], \eta(x) < -L^{3/4}) = O(\exp(-L^{1/2})). \tag{A.3}$$

The dynamics in countinuous time can then be obtained by considering $(\tau_n)_{n\geqslant 0}$ a Poisson clock process of intensity L and setting $\widetilde\eta^2(t) = \widehat\eta(n)$ if $t \in [\tau_n, \tau_{n+1})$. Then one can conclude by remarking that with high probability

$$\tau_{\exp(2L^{1/4})} \geqslant \exp(L^{1/4}). \tag{A.4}$$

$\square$

The second result concerns estimate on $\widetilde{f}^\delta$, the solution of (1.3) with initial condition f_0^δ instead of f_0.

Lemma A.2. *For $t \leqslant t_0(\bar\delta)$,*

$$a(\widetilde{f}^\delta(\cdot,t)) \leqslant a(\widetilde{f}_0^\delta) - 2t + e^{-\frac{\bar\delta^2}{16t}}. \tag{A.5}$$

Moreover, uniformly on $x \leqslant (-1 + \bar\delta^2/2)$

$$\widetilde{f}^\delta(x,t) \geqslant (x+1)(1 - e^{-\frac{\bar\delta^2}{16t}}). \tag{A.6}$$

and

$$\widetilde{f}^\delta(x,t) \geqslant \bar\delta/8, \quad \forall x \in (-1+\bar\delta/4, 1-\bar\delta/4). \tag{A.7}$$

Proof. First notice that

$$a(\widetilde{f}^{\delta}(\cdot,t)) = \partial_x \widetilde{f}^{\delta}(-1,t) - \partial_x \widetilde{f}^{\delta}(1,t) = 2\partial_x \widetilde{f}^{\delta}(-1,t). \tag{A.8}$$

Thus (A.5) can be proved if one controls $\partial_x \widetilde{f}^{\delta}(-1,t)$ up to time t_0. Note that $\partial_x \widetilde{f}^{\delta}$ is solution of the heat equation in $[-1,1]$ with Neumann boundary condition and thus that

$$\partial_x \widetilde{f}^{\delta}(x,t) := \mathbf{E}_x[\partial_x \widetilde{f}^{\delta}(B_t,t)], \tag{A.9}$$

where $\mathbf{E}_x$ is the expectation with respect to standard Brownian Motion reflected on the boundaries. As the initial derivative is in $[-1,1]$ one has

$$\mathbf{E}_x[\partial_x \widetilde{f}^{\delta}(B_t,t)] \geqslant 1 - 2\mathbf{P}_x[B_t \geqslant\, -1+\bar{\delta}]. \tag{A.10}$$

If t is much smaller than $\bar{\delta}^2$ and $x \leqslant\, -1+\bar{\delta}/2$

$$\mathbf{P}_x[B_t \geqslant\, -1+\bar{\delta}] \leqslant e^{-\frac{\bar{\delta}^2}{16t}}/2. \tag{A.11}$$

The result for $x = -1$ implies (A.5), whereas integrating the result gives (A.6). The last statement can be deduced from the fact that $\widetilde{f}^{\delta}$ is symmetric and non-decreasing on $[-1,1]$.

□

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